Similarity measures of intuitionistic fuzzy sets based on Hausdorff distance

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Abstract

This paper presents a new method for similarity measures between intuitionistic fuzzy sets (IFSs). We will present a method to calculate the distance between IFSs on the basis of the Hausdorff distance. We will then use this distance to generate a new similarity measure to calculate the degree of similarity between IFSs. Finally we will prove some properties of the proposed similarity measure and use several examples to compare the proposed similarity measure with existing methods. Numerical results show that the proposed similarity measure is much simpler than existing methods and is well suited to use with linguistic variables.

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1. Introduction

The notion of defining intuitionistic fuzzy sets (IFSs) for fuzzy set generalizations, introduced by Atanassov (1986), has proven interesting and useful in various application areas. Since this fuzzy set generalization can present the degrees of membership and non-membership with a degree of hesitancy, the knowledge and semantic representation becomes more meaningful and applicable (cf. Atanassov, 1986, 1994, 1999). These IFSs have been widely studied and applied in a variety of areas such as logic programming (cf. Atanassov and Gargov, 1990; Atanassov and Georgeiv, 1993), decision making problems (Szmidt and Kacprzyk, 1996) and in medical diagnostics (De et al., 2001), etc.

In many applications, the similarity between IFSs is very important. Recently, Li and Cheng (2002) discussed similarity measures on IFSs and showed how these measures may be used in pattern recognition problems. However, Li and Cheng’s similarity measures may not be effective in some cases. To overcome the drawbacks of Li and Cheng’s methods, Liang and Shi (2003) proposed several new similarity measures and also discussed the relationships between these measures. Numerical comparisons showed that Liang and
Shi’s similarity measures are more reasoned than Li and Cheng’s. On the other hand, Mitchell (2003) interpreted IFSs as ensembles of ordered fuzzy sets from a statistical viewpoint to modify Li and Cheng’s methods.

In this study, we present a new method to calculate the degree of similarity between IFSs based on the Hausdorff distance concept. Numerical results show that the proposed similarity measure is much simpler than existing methods and is well suited to use with linguistic variables. The organization of this paper is as follows: in Section 2, we review some similarity measures of IFSs (cf. Li and Cheng, 2002; Liang and Shi, 2003; Mitchell, 2003), and the Hausdorff distance concept. In Section 3, we present a method to calculate the distance between IFSs on the basis of the Hausdorff distance. We then use this distance to generate a new similarity measure to calculate the degree of similarity between IFSs. We will also prove some properties of the proposed similarity measure. In Section 4, we compare the proposed similarity measure with existing methods. Conclusions are made in Section 5.

2. Preliminaries

2.1. Intuitionistic fuzzy set

An IFS \( \tilde{A} \) in \( X \) is given by Atanassov (1986) as

\[
\tilde{A} = \{(x, \mu_{\tilde{A}}(x), \nu_{\tilde{A}}(x)) \mid x \in X\},
\]

where

\[
\mu_{\tilde{A}}(x) : X \to [0, 1] \quad \nu_{\tilde{A}}(x) : X \to [0, 1]
\]

with the condition

\[
0 \leq \mu_{\tilde{A}}(x) + \nu_{\tilde{A}}(x) \leq 1 \quad \forall x \in X.
\]

The numbers \( \mu_{\tilde{A}}(x) \) and \( \nu_{\tilde{A}}(x) \) denote the degree of membership and non-membership of \( x \) to \( \tilde{A} \), respectively. In this paper, we denote IFSs(\( X \)) as the set of all IFSs in \( X \). In the following, we present some basic operations on IFSs which will be needed in our next discussion.

**Definition 2.1.** If \( \tilde{A} \) and \( \tilde{B} \) are two IFSs of the set \( X \), then

(i) \( \tilde{A} \subseteq \tilde{B} \) if and only if \( \forall x \in X, \mu_{\tilde{A}}(x) \leq \mu_{\tilde{B}}(x) \) and \( \nu_{\tilde{A}}(x) \geq \nu_{\tilde{B}}(x) \);

(ii) \( \tilde{A} = \tilde{B} \) if and only if \( \forall x \in X, \mu_{\tilde{A}}(x) = \mu_{\tilde{B}}(x) \) and \( \nu_{\tilde{A}}(x) = \nu_{\tilde{B}}(x) \).

Measuring the similarity between IFSs is important in pattern recognition research. Some methods have previously been advanced to calculate the degree of similarity between IFSs (Li and Cheng, 2002; Liang and Shi, 2003; Mitchell, 2003). In the following, we shall review these similarity measures. In the study of the similarity between IFSs, Li and Cheng (2002) introduced the following definition.

**Definition 2.2.** A mapping \( S : \text{IFSs}(X) \times \text{IFSs}(X) \to [0, 1] \), \( S(\tilde{A}, \tilde{B}) \) is said to be the degree of similarity between \( \tilde{A} \in \text{IFSs}(X) \) and \( \tilde{B} \in \text{IFSs}(X) \), if \( S(\tilde{A}, \tilde{B}) \) satisfies the properties (P1–P4):

(P1) \( 0 \leq S(\tilde{A}, \tilde{B}) \leq 1 \)

(P2) \( S(\tilde{A}, \tilde{B}) = 1 \) if \( \tilde{A} = \tilde{B} \)

(P3) \( S(\tilde{A}, \tilde{B}) = S(\tilde{B}, \tilde{A}) \)

(P4) \( S(\tilde{A}, \tilde{C}) \leq S(\tilde{C}, \tilde{B}) \) and \( S(\tilde{C}, \tilde{A}) \leq S(\tilde{B}, \tilde{C}) \) if \( \tilde{A} \subseteq \tilde{B} \subseteq \tilde{C}, \tilde{C} \in \text{IFSs}(X) \).

**Remark.** For applications involving pattern recognition Mitchell (2003) suggested replacing (P2) with a new “strong” version (P2‘):

(P2’) \( S(\tilde{A}, \tilde{B}) = 1 \) if and only if \( \tilde{A} = \tilde{B} \).

Assume that there are two IFSs \( \tilde{A} \) and \( \tilde{B} \) in \( X = \{x_1, \ldots, x_n\} \), the degree of similarity between the two IFSs, \( \tilde{A} \) and \( \tilde{B} \), can then be calculated as follows (Li and Cheng, 2002):

\[
S_0^p(\tilde{A}, \tilde{B}) = 1 - \frac{1}{\sqrt{n}} \sqrt{\sum_{i=1}^{n} [\mu_{\tilde{A}}(i) - \mu_{\tilde{B}}(i)]^p},
\]

where \( \mu_{\tilde{A}}(i) = (\mu_{\tilde{A}}(x_i) + 1 - \nu_{\tilde{A}}(x_i))/2, \nu_{\tilde{A}}(i) = (\mu_{\tilde{B}}(x_i) + 1 - \nu_{\tilde{B}}(x_i))/2 \) and \( 1 \leq p < \infty \).

To overcome the drawbacks of \( S_0^p(\tilde{A}, \tilde{B}) \), Liang and Shi (2003) proposed the following similarity measures between IFSs. Let \( \phi_{A \tilde{B}}(i) = |\mu_{\tilde{A}}(x_i) - \mu_{\tilde{B}}(x_i)|/(1 - \nu_{\tilde{B}}(x_i))/2 \) and \( \phi_{A \tilde{B}}(i) = (1 - \nu_{\tilde{A}}(x_i))/2 - (1 - \nu_{\tilde{B}}(x_i))/2 \). The degree of similarity between the
two IFSs, \( \tilde{A} \) and \( \tilde{B} \), can then be calculated as follows:

\[
S_e^p(\tilde{A}, \tilde{B}) = \frac{1}{1 - \frac{1}{\sqrt{n}} \sqrt{\frac{1}{n} \sum_{i=1}^{n} (\phi_{\tilde{A},\tilde{B}}(i) + \phi_{\tilde{B},\tilde{A}}(i))^p}}.
\]

(2)

To get more information on IFSs, Liang and Shi (2003) considered another definition of similarity measures between \( A \) and \( B \). Let

\[
m_{\tilde{A}1}(i) = \frac{\mu_{A}(x_i) + m_{\tilde{A}1}(i)}{2},
\]

\[
m_{\tilde{A}2}(i) = \frac{m_{\tilde{A}}(i) + 1 - v_{\tilde{A}}(x_i)}{2},
\]

\[
m_{\tilde{B}1}(i) = \frac{\mu_{B}(x_i) + m_{\tilde{B}1}(i)}{2},
\]

\[
m_{\tilde{B}2}(i) = \frac{m_{\tilde{B}}(i) + 1 - v_{\tilde{B}}(x_i)}{2}.
\]

We can then calculate the degree of similarity between the two IFSs, \( \tilde{A} \) and \( \tilde{B} \), as follows:

\[
S_e^p(\tilde{A}, \tilde{B}) = 1 - \frac{1}{\sqrt{n}} \sqrt{\frac{1}{n} \sum_{i=1}^{n} (\phi_{\tilde{A}1}(i) + \phi_{\tilde{B}2}(i))^p},
\]

(3)

where \( \phi_{\tilde{A}1}(i) = |m_{\tilde{A}1}(i) - m_{\tilde{B}1}(i)|/2 \) and \( \phi_{\tilde{B}2}(i) = |m_{\tilde{B}2}(i) - m_{\tilde{A}2}(i)|/2 \). Finally, they considered the following similarity measure to obtain all available information on IFSs. Let \( \phi_1(i) = \phi_{\tilde{A}1}(i) + \phi_{\tilde{B}2}(i) \) or \( \phi_1(i) = \phi_{\tilde{A}2}(i) + \phi_{\tilde{B}1}(i) \), \( \phi_2(i) = |m_{\tilde{A}}(i) - m_{\tilde{B}}(i)| \), \( \ell_{\tilde{A}}(i) = (1 - v_{\tilde{A}}(x_i) - \mu_{\tilde{A}}(x_i))/2 \), \( \ell_{\tilde{B}}(i) = (1 - v_{\tilde{B}}(x_i) - \mu_{\tilde{B}}(x_i))/2 \) and let \( \phi_3(i) \) be defined as

\[
\phi_3(i) = \max\{\ell_{\tilde{A}}(i), \ell_{\tilde{B}}(i)\} - \min\{\ell_{\tilde{A}}(i), \ell_{\tilde{B}}(i)\}.
\]

The degree of similarity between the two IFSs, \( \tilde{A} \) and \( \tilde{B} \), can then be calculated as follows:

\[
S_h^p(\tilde{A}, \tilde{B}) = 1 - \frac{1}{\sqrt{n}} \sqrt{\frac{1}{n} \sum_{i=1}^{n} \left( \sum_{m=1}^{3} \omega_m \phi_m(i) \right)^p},
\]

(4)

where \( 0 \leq \omega_m \leq 1 \), \( \sum_{m=1}^{3} \omega_m = 1 \).

Mitchell (2003) adopted a statistical approach and interpreted IFSs as ensembles of ordered fuzzy sets to modify Li and Cheng’s similarity measures. Let \( \rho_\mu(\tilde{A}, \tilde{B}) \) and \( \rho_\nu(\tilde{A}, \tilde{B}) \) denote the similarity measures between the “low” membership functions \( \mu_{\tilde{A}} \) and \( \mu_{\tilde{B}} \) and between the “high” membership functions \( 1 - v_{\tilde{A}} \) and \( 1 - v_{\tilde{B}} \), respectively, as follows:

\[
\rho_\mu(\tilde{A}, \tilde{B}) = S(\mu_{\tilde{A}}, \mu_{\tilde{B}})
\]

\[
= 1 - \frac{1}{\sqrt{n}} \sqrt{\frac{1}{n} \sum_{i=1}^{n} |\mu_{\tilde{A}}(x_i) - \mu_{\tilde{B}}(x_i)|^p},
\]

\[
\rho_\nu(\tilde{A}, \tilde{B}) = S(1 - v_{\tilde{A}}, 1 - v_{\tilde{B}})
\]

\[
= 1 - \frac{1}{\sqrt{n}} \sqrt{\frac{1}{n} \sum_{i=1}^{n} |v_{\tilde{A}}(x_i) - v_{\tilde{B}}(x_i)|^p}.
\]

They then defined the modified similarity measure between \( \tilde{A} \) and \( \tilde{B} \) with

\[
S_{\text{mod}}(\tilde{A}, \tilde{B}) = \frac{1}{2}(\rho_\mu(\tilde{A}, \tilde{B}) + \rho_\nu(\tilde{A}, \tilde{B})).
\]

(5)

2.2. Hausdorff distance

The Hausdorff distance (Nadler, 1978) is a measure of how much two non-empty compact (closed and bounded) sets \( A \) and \( B \) in a metric space \( S \) resemble each other with respect to their positions. Let \( d(a, b) \) be a metric for \( S \). We adopt the notation and write

\[
d(z, A) = \min\{d(z, a) \mid a \in A\}.
\]

The “nonsymmetric” or “one-way” Hausdorff measure is

\[
H^+(A, B) = \max_{a \in A} d(a, B).
\]

The Hausdorff distance \( H(A, B) \) is defined (Nadler, 1978) by

\[
H(A, B) = \max\{H^+(A, B), H^+(B, A)\}.
\]

Let us consider the real space \( \mathbb{R} \). For any two intervals \( A = [a_1, a_2] \) and \( B = [b_1, b_2] \), the Hausdorff distance \( H(A, B) \) is given by

\[
H(A, B) = \max\{|a_1 - b_1|, |a_2 - b_2|\}.
\]

(6)

3. Similarity measures between intuitionistic fuzzy sets

Distance is an important concept in science and engineering. In the following, we define a distance between two IFSs based on the Hausdorff
distance. Let $\tilde{A}$ and $\tilde{B}$ be two IFSs in $X = \{x_1, x_2, \ldots, x_n\}$ and let $I_\tilde{A}(x_i)$ and $I_\tilde{B}(x_i)$ be subintervals on $[0, 1]$ denoted by the following:

$$I_\tilde{A}(x_i) = [\mu_\tilde{A}(x_i), 1 - v_\tilde{A}(x_i)],$$
$$I_\tilde{B}(x_i) = [\mu_\tilde{B}(x_i), 1 - v_\tilde{B}(x_i)], \quad i = 1, 2, \ldots, n.$$

Let $H(I_\tilde{A}(x_i), I_\tilde{B}(x_i))$ be the Hausdorff distance between $I_\tilde{A}(x_i)$ and $I_\tilde{B}(x_i)$. We then can define the distance $d_H(\tilde{A}, \tilde{B})$ between $\tilde{A}$ and $\tilde{B}$ as follows:

$$d_H(\tilde{A}, \tilde{B}) = \frac{1}{n} \sum_{i=1}^{n} H(I_\tilde{A}(x_i), I_\tilde{B}(x_i)).$$

It is natural to then ask “Is the defined distance $d_H(\tilde{A}, \tilde{B})$ reasonable?” We answer this question in Proposition 3.1 following.

**Proposition 3.1.** The defined distance $d_H(\tilde{A}, \tilde{B})$ between IFSs $\tilde{A}$ and $\tilde{B}$ satisfies the following properties (D1–D4):

(D1) $0 \leq d_H(\tilde{A}, \tilde{B}) \leq 1$;

(D2) $\tilde{A} = \tilde{B}$ if and only if $d_H(\tilde{A}, \tilde{B}) = 0$;

(D3) $d_H(\tilde{A}, \tilde{B}) = d_H(\tilde{B}, \tilde{A})$;

(D4) If $\tilde{A} \subseteq \tilde{B} \subseteq \tilde{C}$, $\tilde{A}, \tilde{B}, \tilde{C} \in \text{IFSs}(X)$, then $d_H(\tilde{A}, \tilde{B}) \leq d_H(\tilde{A}, \tilde{C})$ and $d_H(\tilde{B}, \tilde{C}) \leq d_H(\tilde{A}, \tilde{C})$.

**Proof.** It is easy to see that $d(\tilde{A}, \tilde{B})$ satisfies the properties (D1)–(D3). We therefore only prove (D4). Let $\tilde{A} \subseteq \tilde{B} \subseteq \tilde{C}$, then $\mu_\tilde{A}(x_i) \leq \mu_\tilde{B}(x_i) \leq \mu_\tilde{C}(x_i)$, and $v_\tilde{A}(x_i) \geq v_\tilde{B}(x_i) \geq v_\tilde{C}(x_i)$, $\forall x_i \in X$. By (6), we have

$$H(I_{\tilde{A}}(x_i), I_{\tilde{C}}(x_i)) = \max\{\mu_{\tilde{A}}(x_i) - \mu_{\tilde{C}}(x_i), |v_{\tilde{A}}(x_i) - v_{\tilde{C}}(x_i)|\},$$

$$H(I_{\tilde{A}}(x_i), I_{\tilde{B}}(x_i)) = \max\{\mu_{\tilde{A}}(x_i) - \mu_{\tilde{B}}(x_i), |v_{\tilde{A}}(x_i) - v_{\tilde{B}}(x_i)|\},$$

$$H(I_{\tilde{B}}(x_i), I_{\tilde{C}}(x_i)) = \max\{\mu_{\tilde{B}}(x_i) - \mu_{\tilde{C}}(x_i), |v_{\tilde{B}}(x_i) - v_{\tilde{C}}(x_i)|\}.$$

(i) If $|\mu_{\tilde{A}}(x_i) - \mu_{\tilde{C}}(x_i)| \geq |v_{\tilde{A}}(x_i) - v_{\tilde{C}}(x_i)|$ then

$$H(I_{\tilde{A}}(x_i), I_{\tilde{C}}(x_i)) = |\mu_{\tilde{A}}(x_i) - \mu_{\tilde{C}}(x_i)|.$$ But we have

$$|v_{\tilde{A}}(x_i) - v_{\tilde{B}}(x_i)| \leq |v_{\tilde{A}}(x_i) - v_{\tilde{C}}(x_i)|,$$

$$|v_{\tilde{B}}(x_i) - v_{\tilde{C}}(x_i)| \leq |v_{\tilde{A}}(x_i) - v_{\tilde{C}}(x_i)|,$$

$$|\mu_{\tilde{A}}(x_i) - \mu_{\tilde{C}}(x_i)| \leq |\mu_{\tilde{A}}(x_i) - \mu_{\tilde{C}}(x_i)|.$$ (8)

(ii) From (i) and (ii), we have

$$|\mu_{\tilde{A}}(x_i) - \mu_{\tilde{C}}(x_i)| \leq |\mu_{\tilde{A}}(x_i) - \mu_{\tilde{C}}(x_i)|$$

and

$$|\mu_{\tilde{B}}(x_i) - \mu_{\tilde{C}}(x_i)| \leq |\mu_{\tilde{A}}(x_i) - \mu_{\tilde{C}}(x_i)|.$$ (10)

Combining (8)–(10), we obtain

$$H(I_{\tilde{A}}(x_i), I_{\tilde{C}}(x_i)) \leq H(I_{\tilde{A}}(x_i), I_{\tilde{C}}(x_i))$$

and

$$H(I_{\tilde{B}}(x_i), I_{\tilde{C}}(x_i)) \leq H(I_{\tilde{B}}(x_i), I_{\tilde{C}}(x_i)).$$

It follows that $d_H(\tilde{A}, \tilde{B}) \leq d_H(\tilde{A}, \tilde{C})$ and $d_H(\tilde{B}, \tilde{C}) \leq d_H(\tilde{A}, \tilde{C})$.

On the other hand, we have

$$|\mu_{\tilde{A}}(x_i) - \mu_{\tilde{B}}(x_i)| \leq |\mu_{\tilde{A}}(x_i) - \mu_{\tilde{C}}(x_i)|$$

and

$$|\mu_{\tilde{B}}(x_i) - \mu_{\tilde{C}}(x_i)| \leq |\mu_{\tilde{A}}(x_i) - \mu_{\tilde{C}}(x_i)|.$$ (11)

Combining (11)–(13), we obtain

$$H(I_{\tilde{A}}(x_i), I_{\tilde{C}}(x_i)) \leq H(I_{\tilde{A}}(x_i), I_{\tilde{C}}(x_i))$$

and

$$H(I_{\tilde{B}}(x_i), I_{\tilde{C}}(x_i)) \leq H(I_{\tilde{A}}(x_i), I_{\tilde{C}}(x_i)).$$

It follows that $d_H(\tilde{A}, \tilde{B}) \leq d_H(\tilde{A}, \tilde{C})$ and $d_H(\tilde{B}, \tilde{C}) \leq d_H(\tilde{A}, \tilde{C})$. From (i) and (ii), we can obtain the property (D4). □

It is well known that distance measures and similarity measure are dual concepts. Therefore,
we may use the distance measure to define a similarity measure. Let \( f \) be a monotone decreasing function. Since \( 0 \leq d_{if}(\bar{A}, \bar{B}) \leq 1 \)
\[
f(1) \leq f(d_{if}(\bar{A}, \bar{B})) \leq f(0).
\]
This implies
\[
0 \leq \frac{f(d_{if}(\bar{A}, \bar{B})) - f(1)}{f(0) - f(1)} \leq 1.
\]
Thus, we may define the similarity measure between IFSs \( \bar{A} \) and \( \bar{B} \) as follows:
\[
S(\bar{A}, \bar{B}) = \frac{f(d_{if}(\bar{A}, \bar{B})) - f(1)}{f(0) - f(1)}.
\]
(14)

According to Proposition 3.1, \( S(\bar{A}, \bar{B}) \) satisfies properties (P1)–(P4). From (D2), \( S(\bar{A}, \bar{B}) \) also satisfies the property (P2*).

The problem here is to select a useful and reasonable \( f \). The simplest \( f \) is chosen as
\[
f(x) = 1 - x.
\]
Then a similarity measure between \( \bar{A} \) and \( \bar{B} \) is denoted as follows:
\[
S_e(\bar{A}, \bar{B}) = 1 - d_{if}(\bar{A}, \bar{B}).
\]
(15)

It is well known that an exponential operation is highly useful in dealing with a similarity relation (cf. Zadeh, 1971), Shannon entropy (cf. Pal and Pal, 1991, 1992) and in cluster analysis (cf. Wu and Yang, 2002; Yang and Wu, 2004). We therefore choose
\[
f(x) = e^{-x}.
\]
Then a similarity measure between \( \bar{A} \) and \( \bar{B} \) is defined as follows:
\[
S_e(\bar{A}, \bar{B}) = \frac{e^{-d_{if}(\bar{A}, \bar{B})} - e^{-1}}{1 - e^{-1}}.
\]
(16)

On the other hand, we may choose
\[
f(x) = \frac{1}{1 + x}.
\]
Then a similarity measure between \( \bar{A} \) and \( \bar{B} \) can be defined as follows:
\[
S_e(\bar{A}, \bar{B}) = \frac{1 - d_{if}(\bar{A}, \bar{B})}{1 + d_{if}(\bar{A}, \bar{B})}.
\]
(17)

When the universe of discourse is continuous, say \( X = [a,b] \), we can obtain the following similar result. For IFSs
\[
\tilde{A} = \{(x, \mu_\tilde{A}(x), \nu_\tilde{A}(x)) | x \in [a,b]\}
\]
and
\[
\tilde{B} = \{(x, \mu_\tilde{B}(x), \nu_\tilde{B}(x)) | x \in [a,b]\},
\]
we define a distance measure \( d_{HC} \) as
\[
d_{HC}(\tilde{A}, \tilde{B}) = \frac{1}{b - a} \int_a^b H(I_\tilde{A}(x), I_\tilde{B}(x)) \, dx,
\]
(18)

where
\[
I_\tilde{A}(x) = [\mu_\tilde{A}(x), 1 - \nu_\tilde{A}(x)]
\]
and
\[
I_\tilde{B}(x) = [\mu_\tilde{B}(x), 1 - \nu_\tilde{B}(x)], \quad x \in [a,b].
\]

However, the elements in the universe may have a different importance in pattern recognition. We need to consider the weight of the element so that we have the following weighted distance between IFSs. Assume that the weight of \( x_i \in X \) is \( w_i \), where \( X = \{x_1, x_2, \ldots, x_n\}, \ 0 \leq w_i \leq 1, \ i = 1, 2, \ldots, n \) and \( \sum_{i=1}^n w_i = 1 \). Then the weighted distance between IFSs \( \bar{A} \) and \( \bar{B} \) is defined as:
\[
d_{HW}(\tilde{A}, \tilde{B}) = \sum_{i=1}^n w_i H(I_\tilde{A}(x_i), I_\tilde{B}(x_i)).
\]
(19)

Obviously, if \( w_i = 1/n, \ i = 1, 2, \ldots, n \), Eq. (19) becomes Eq. (7). Thus, Eq. (7) is only a special case of Eq. (19). Similarly, assume that the weight of \( x \in X = [a,b] \) is \( w(x) \), where \( 0 \leq w(x) \leq 1, \int_a^b w(x) \, dx = 1 \). Then the weighted distance between IFSs \( \bar{A} \) and \( \bar{B} \) is
\[
d_{HCW}(\tilde{A}, \tilde{B}) = \int_a^b w(x) H(I_\tilde{A}(x), I_\tilde{B}(x)) \, dx.
\]
(20)

It is obvious that Eq. (20) becomes Eq. (18) if \( w(x) = 1/(b-a) \) for any \( x \in [a,b] \). Thus, Eq. (18) is only a special case of Eq. (20). Using \( d_{HC}(\tilde{A}, \tilde{B}) \), \( d_{HW}(\tilde{A}, \tilde{B}) \) and \( d_{HCW}(\tilde{A}, \tilde{B}) \) to replace \( d_{if}(\bar{A}, \bar{B}) \) in Eq. (14), we can obtain the similarity measures between \( \bar{A} \) and \( \bar{B} \).
4. Numerical examples

In this section, we present several examples to compare the proposed similarity measure with existing similarity measures. For convenience, we consider \( p = 1 \) and \( q_i = \frac{1}{3}, \ i = 1, 2, 3 \) in similarity measures \( S_{d}^{p}, S_{e}^{p}, S_{d}^{p}, S_{h}^{p} \) and \( S_{mod} \).

**Example 1** (Liang and Shi, 2003). Assume that there are three patterns denoted with IFSs in \( X = \{x_1, x_2, x_3\} \). The three patterns are denoted as follows:

\[
\tilde{A}_1 = \{(x_1, 0.3, 0.3), (x_2, 0.2, 0.2), (x_3, 0.1, 0.1)\};
\]

\[
\tilde{A}_2 = \{(x_1, 0.2, 0.2), (x_2, 0.2, 0.2), (x_3, 0.2, 0.2)\};
\]

\[
\tilde{A}_3 = \{(x_1, 0.4, 0.4), (x_2, 0.4, 0.4), (x_3, 0.4, 0.4)\}.
\]

Assume that a sample \( \tilde{B} = \{(x_1, 0.3, 0.3), (x_2, 0.2, 0.2), (x_3, 0.1, 0.1)\} \) is given.

By (1)–(5), we have 

\[
S_{d}^{p}(\tilde{A}_1, \tilde{B}) = S_{d}^{p}(\tilde{A}_2, \tilde{B}) = S_{d}^{p}(\tilde{A}_3, \tilde{B}) = 1;
\]

\[
S_{e}^{1}(\tilde{A}_1, \tilde{B}) = 1, \quad S_{e}^{1}(\tilde{A}_2, \tilde{B}) = 0.933,
\]

\[
S_{e}^{1}(\tilde{A}_3, \tilde{B}) = 0.800;
\]

\[
S_{e}^{i}(\tilde{A}_1, \tilde{B}) = 1, \quad S_{e}^{i}(\tilde{A}_2, \tilde{B}) = 0.967,
\]

\[
S_{e}^{i}(\tilde{A}_3, \tilde{B}) = 0.900;
\]

\[
S_{h}^{1}(\tilde{A}_1, \tilde{B}) = 1, \quad S_{h}^{1}(\tilde{A}_2, \tilde{B}) = 0.956,
\]

\[
S_{h}^{1}(\tilde{A}_3, \tilde{B}) = 0.867;
\]

\[
S_{mod}(\tilde{A}_1, \tilde{B}) = 1, \quad S_{mod}(\tilde{A}_2, \tilde{B}) = 0.933,
\]

\[
S_{mod}(\tilde{A}_3, \tilde{B}) = 0.800.
\]

On the other hand, by (15)–(17), we have 

\[
S_{e}(\tilde{A}_1, \tilde{B}) = 1, \quad S_{e}(\tilde{A}_2, \tilde{B}) = 0.933,
\]

\[
S_{e}(\tilde{A}_3, \tilde{B}) = 0.800;
\]

\[
S_{e}(\tilde{A}_1, \tilde{B}) = 1, \quad S_{e}(\tilde{A}_2, \tilde{B}) = 0.898,
\]

\[
S_{e}(\tilde{A}_3, \tilde{B}) = 0.713;
\]

\[
S_{e}(\tilde{A}_1, \tilde{B}) = 1, \quad S_{e}(\tilde{A}_2, \tilde{B}) = 0.875,
\]

\[
S_{e}(\tilde{A}_3, \tilde{B}) = 0.667.
\]

From this data, it is evident that \( \tilde{A}_1 = \tilde{B} \). That is, the sample \( \tilde{B} \) belongs to the pattern \( \tilde{A}_1 \). The proposed similarity measures together with those of Liang and Shi (2003) and Mitchell (2003) show the correct classification according to the principle of maximum membership degree. But, Li and Cheng’s similarity measures cannot be used to classify this sample.

**Example 2** (Liang and Shi, 2003). Assume that there are two patterns denoted with IFSs in \( X = \{x_1, x_2, x_3\} \). The two patterns are denoted as follows:

\[
\tilde{A}_1 = \{(x_1, 0.2, 0.2), (x_2, 0.2, 0.2), (x_3, 0.2, 0.2)\};
\]

\[
\tilde{A}_2 = \{(x_1, 0.4, 0.4), (x_2, 0.4, 0.4), (x_3, 0.4, 0.4)\}.
\]

Assume that a sample \( \tilde{B} = \{(x_1, 0.3, 0.3), (x_2, 0.3, 0.3), (x_3, 0.1, 0.3)\} \) is given.

By (1)–(5), we have 

\[
S_{d}^{1}(\tilde{A}_1, \tilde{B}) = S_{d}^{1}(\tilde{A}_2, \tilde{B}) = 0.967;
\]

\[
S_{e}^{1}(\tilde{A}_1, \tilde{B}) = 0.900, \quad S_{e}^{1}(\tilde{A}_2, \tilde{B}) = 0.867;
\]

\[
S_{e}^{1}(\tilde{A}_1, \tilde{B}) = 0.933,
\]

\[
S_{e}^{1}(\tilde{A}_2, \tilde{B}) = 0.900;
\]

\[
S_{mod}(\tilde{A}_1, \tilde{B}) = 0.900, \quad S_{mod}(\tilde{A}_2, \tilde{B}) = 0.867.
\]

On the other hand, by (15)–(17), we have 

\[
S_{e}(\tilde{A}_1, \tilde{B}) = 0.900, \quad S_{e}(\tilde{A}_2, \tilde{B}) = 0.833;
\]

\[
S_{e}(\tilde{A}_1, \tilde{B}) = 0.849, \quad S_{e}(\tilde{A}_2, \tilde{B}) = 0.757;
\]

\[
S_{e}(\tilde{A}_1, \tilde{B}) = 0.818, \quad S_{e}(\tilde{A}_2, \tilde{B}) = 0.714.
\]

Based on the above results, it is seen that the sample \( \tilde{B} \) belongs to the pattern \( \tilde{A}_1 \) according to the principle of the maximum degree of similarity between IFSs. However, the similarity measures \( S_{d}^{p} \) and \( S_{e}^{p} \) cannot classify this sample.
Example 3 (Liang and Shi, 2003). Assume that there are three patterns denoted with IFSs in $X = \{x_1, x_2, x_3\}$. The three patterns are denoted as follows:

$\tilde{A}_1 = \{(x_1, 0.1, 0.1), (x_2, 0.5, 0.1), (x_3, 0.1, 0.9)\}$;

$\tilde{A}_2 = \{(x_1, 0.5, 0.5), (x_2, 0.7, 0.3), (x_3, 0.0, 0.8)\}$;

$\tilde{A}_3 = \{(x_1, 0.7, 0.2), (x_2, 0.1, 0.8), (x_3, 0.4, 0.4)\}$.

Assume that a sample $\tilde{B} = \{(x_1, 0.4, 0.4), (x_2, 0.6, 0.2), (x_3, 0.0, 0.8)\}$ is given.

By (1)–(5), we have

$S^c_1(\tilde{A}_1, \tilde{B}) = S^c_1(\tilde{A}_2, \tilde{B}) = 1$, $S^c_1(\tilde{A}_3, \tilde{B}) = 0.600$;

$S^a_1(\tilde{A}_1, \tilde{B}) = 0.833$, $S^a_1(\tilde{A}_2, \tilde{B}) = 0.933$, $S^a_1(\tilde{A}_3, \tilde{B}) = 0.600$;

$S^l_1(\tilde{A}_1, \tilde{B}) = 0.917$, $S^l_1(\tilde{A}_2, \tilde{B}) = 0.967$, $S^l_1(\tilde{A}_3, \tilde{B}) = 0.600$;

$S^l_h(\tilde{A}_1, \tilde{B}) = 0.889$, $S^l_h(\tilde{A}_2, \tilde{B}) = 0.956$, $S^l_h(\tilde{A}_3, \tilde{B}) = 0.722$;

$S_{mod}(\tilde{A}_1, \tilde{B}) = 0.833$, $S_{mod}(\tilde{A}_2, \tilde{B}) = 0.933$, $S_{mod}(\tilde{A}_3, \tilde{B}) = 0.600$.

On the other hand, by (15)–(17), we have

$S_c(\tilde{A}_1, \tilde{B}) = 0.833$, $S_c(\tilde{A}_2, \tilde{B}) = 0.933$, $S_c(\tilde{A}_3, \tilde{B}) = 0.567$;

$S_c(\tilde{A}_1, \tilde{B}) = 0.757$, $S_c(\tilde{A}_2, \tilde{B}) = 0.898$, $S_c(\tilde{A}_3, \tilde{B}) = 0.444$;

$S_c(\tilde{A}_1, \tilde{B}) = 0.714$, $S_c(\tilde{A}_2, \tilde{B}) = 0.875$, $S_c(\tilde{A}_3, \tilde{B}) = 0.395$.

It is seen that the sample $\tilde{B}$ belongs to the pattern $\tilde{A}_2$ according to the principle of the maximum degree of similarity between IFSs. However, the similarity measures $S^p_0$ cannot classify this sample.

From Examples 1–3, we see that the proposed similarity measure is as good as $S^p_0$, $S^c_0$ and $S_{mod}$.
The results are summarized in Table 1. The following abbreviated notations are used in Table 1.

L. LARGE
M.L.L. More or less LARGE
V.L. Very LARGE
V.V.L. Very very LARGE

From the viewpoint of mathematical operations, the similarities between the above IFSs have the following requirements:

\[ S(L, M.L.L.) > S(L, V.L.) \]
\[ > S(L, V.V.L.) \]  \hspace{1cm} (21)

\[ S(M.L.L., L.) > S(M.L.L., V.L.) \]
\[ > S(M.L.L., V.V.L.) \]  \hspace{1cm} (22)

\[ S(V.L., V.V.L.) > S(V.L., L.) \]
\[ > S(V.L., M.L.L.) \]  \hspace{1cm} (23)

\[ S(V.V.L., V.L.) > S(V.V.L., L.) \]
\[ > S(V.V.L., M.L.L.) \]  \hspace{1cm} (24)

From Table 1, we see that the proposed similarity measures satisfy requirements (21)–(24). Therefore, the proposed similarity measures are reliable in applications with compound linguistic variables.

5. Conclusions

In this paper, we presented a new method for similarity measures between IFSs. First, we gave a method to calculate the distance of IFSs on the basis of the Hausdorff distance. We then use this distance to generate a new similarity measure to calculate the degree of similarity between IFSs. We not only proved some properties of the proposed similarity measure, we also used several examples to make comparisons between the proposed similarity measure and existing methods. Based on the results, we see that the proposed similarity measure is much simpler than the existing methods and is well suited to use with linguistic variables.

References


